Modified Spectral Boundary conditions in the Bag Model

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Abstract. We propose a reduced form of Atiah-Patodi-Singer spectral boundary conditions for odd (d) dimensional spatial bag evolving in even ($d+1$) dimensional space-time. The modified boundary conditions are manifestly chirally invariant and do not depend on time. This allows to apply Hamiltonian approach to confined massless fermions and study chirality effects in spatially closed volume. The modified boundary conditions are equally suitable for chiral fermions in Minkowski and Euclidean metric space-times.

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Introduction

The two principal problems of QCD are confinement and spontaneous breaking of chiral invariance. Both phenomena take place in the strongly interacting domain where the theory becomes nonperturbative. Most probably they are interrelated. However, usually they were considered separately. Up to now the spontaneous chiral invariance breaking (SCIB) was discussed mostly in the infinite space. It would be interesting to study specific features of SCIB that appear due to localization of quarks in finite volume. In order to do that one needs a chiral invariant model of confinement.

There exists a rich family of bag models. The first was the famous MIT bag [1] that successfully reproduced the spectrum and many features of hadrons. A generalization of the MIT model are so-called chiral bags [2, 3]. An apparent drawback of these models is that the boundary conditions are explicitly chirally noninvariant.

Attempts to save the situation led to the so called cloudy bag model [4] where the chiral symmetry was restored by pion emission from the bag surface (the pion cloud). But this model is sensitive to details of the adopted scheme of quark-pion interaction. Thus neither of the listed models is suited to the discussion of SCIB in finite volume.

A way to lock fermions in finite volume without spoiling the chiral symmetry is to impose the so-called **spectral boundary conditions** (SBC). They were first

introduced by Atiah, Patodi and Singer (APS) who investigated anomalies on manifolds with boundaries [5]. Later these boundary conditions were widely applied in studies of index theorems on various manifolds [6].

Unlike the already mentioned ones the APS conditions are nonlocal. They are defined on the boundary as a whole. This looks natural for finite Euclidean manifolds but is inconvenient for physical models where the time evolution takes place. The evolution converts the spatial boundary of static physical bag into an infinite space-time cylinder. Constraining fields on the entire *world cylinder* including both "the past" and "the future" contradicts causality and complicates generalization to Minkowski space.

In order to avoid this difficulty we propose a purely spatial version of spectral boundary conditions. These modified conditions do not depend on time and, therefore, are acceptable from the physical point of view. Besides, they make possible the usual Hamiltonian description of the system and may be used in Minkowski space-time.

The paper has the following structure. We shall review the classical APS boundary conditions in Section 1. In Section 2 we shall formulate the modified spectral conditions and discuss their properties. At the end we shall summarize the results and mention the prospects.

1. The APS boundary conditions and their physics

1.1. Conventions

We will start from the traditional form of SBC. First we will introduce coordinates, Dirac matrices and the gauge that allow to most clearly define the spectral boundary conditions. For simplicity we will consider the 4-dimensional case. The generalization to higher even dimensions is straightforward.

Let us consider massless fermions interacting with gauge field $\hat{A}$ in a closed Euclidean domain B_4. We choose the curvilinear coordinates so that near the boundary ∂B_4 the first coordinate ξ points along the outward normal while the three others, q^i, parametrize ∂B_4 itself. The origin $\xi = 0$ lies on ∂B_4. For simplicity we shall assume that near the surface the metric $g_{\alpha\beta}$ depends only on q so that

$$ds^2 = d\xi^2 + g_{ik}(q)\, dq^i\, dq^k. \tag{1}$$

Moreover, we choose the gauge so that on the boundary $\hat{A}_\xi = 0$.

Now we must fix the Dirac matrix γ^ξ. Let I be the 2×2 unity matrix. Then

$$\gamma^\xi = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}; \qquad \gamma^q = \begin{pmatrix} 0 & \sigma^q \\ \sigma^q & 0 \end{pmatrix}. \tag{2}$$

Matrices σ^q are the ordinary Pauli σ-matrices. With these definitions the Dirac operator of massless fermions on the surface takes the form,

$$-i\nabla\!\!\!/|_{\partial B_4} = -i\gamma^\alpha \nabla_\alpha = \begin{pmatrix} 0 & \hat{M} \\ \hat{M}^\dagger & 0 \end{pmatrix} = \begin{pmatrix} 0 & I\,\partial_\xi - i\hat{\nabla} \\ -I\,\partial_\xi - i\hat{\nabla} & 0 \end{pmatrix}, \tag{3}$$

where $\hat{\nabla} = \sigma^q \nabla_q$ is the convolution of covariant gradient along the boundary ∇_q with σ-matrices. Note that Hermitian conjugated operators $\hat{M}$ and $\hat{M}^\dagger$ differ only by the sign of ∂_ξ-derivative.

Further on we shall call the covariant derivative $-i\hat{\nabla}$ on the boundary the **boundary operator.** It is a linear differential operator acting on 2-spinors. It is Hermitian and includes tangential gauge field $\hat{A}_q$ and the spin connection which arises from the curvature of ∂B_4.

The massless Dirac operator anticommutes with γ^5-matrix:

$$\left\{-i\nabla\!\!\!\!/,\, \gamma^5\right\} = 0, \qquad \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{4}$$

and gauge interactions do not change helicity of massless quarks. This property is called chiral invariance. In order to preserve it in finite space one needs chirally invariant boundary conditions.

1.2. The APS boundary conditions

1.2.1. The definition Atiah, Patodi and Singer investigated spectra of Dirac operator on manifolds with boundaries. If we separate upper and lower (left and right) components of 4-spinors the corresponding eigenvalue equation for $-i\nabla\!\!\!\!/$ will take the form

$$-\, i\nabla\!\!\!\!/\, \psi_\Lambda = -i\nabla\!\!\!\!/ \begin{pmatrix} u_\Lambda \\ v_\Lambda \end{pmatrix} = \Lambda \begin{pmatrix} u_\Lambda \\ v_\Lambda \end{pmatrix} = \Lambda\, \psi_\Lambda. \tag{5}$$

The next step is to Fourier-expand u and v near the boundary. Let 2-spinors $e_\lambda(q)$ be eigenfunctions of the boundary operator $-i\hat{\nabla}$:

$$-\, i\hat{\nabla}\, e_\lambda(q) = \lambda\, e_\lambda(q). \tag{6}$$

Note that the form of this equation and the eigenfunctions $e_\lambda(q)$ depend on gauge. It is here that the gauge condition $\hat{A}_\xi(0,\, q) = 0$ becomes important.

The operator $-i\hat{\nabla}$ is Hermitian so λ's are real. The functions e_λ form an orthogonal basis on ∂B_4. In principle $-i\hat{\nabla}$ may have zero-modes but sphere and convex manifolds are not the case.

In the vicinity of the boundary spinors u_Λ and v_Λ may be expanded in series in e_λ:

$$u_\Lambda(\xi,\, q) = \sum_\lambda f_\Lambda^\lambda(\xi)\, e_\lambda(q), \quad f_\Lambda^\lambda(\xi) = \int_{\partial B_4} e_\lambda^\dagger(q)\, u_\Lambda(\xi,\, q)\, \sqrt{g}\, d^3q; \tag{7a}$$

$$v_\Lambda(\xi,\, q) = \sum_\lambda g_\Lambda^\lambda(\xi)\, e_\lambda(q), \quad g_\Lambda^\lambda(\xi) = \int_{\partial B_4} e_\lambda^\dagger(q)\, v_\Lambda(\xi,\, q)\, \sqrt{g}\, d^3q; \tag{7b}$$

where $g = \det ||g_{ik}||$ is the determinant of metric on the boundary.

The spectral boundary conditions state that on the boundary, *i. e.* at $\xi = 0$

$$f_\Lambda^\lambda\Big|_{\partial B_4} = 0 \qquad \text{for} \qquad \lambda > 0; \tag{8a}$$

$$g_\Lambda^\lambda\Big|_{\partial B_4} = 0 \qquad \text{for} \qquad \lambda < 0. \tag{8b}$$

Another way to say this is to introduce integral projectors $\mathcal{P}^+$ and $\mathcal{P}^-$ onto boundary modes with positive and negative λ:

$$\mathcal{P}^+(q,\,q') = \sum_{\lambda>0} e_\lambda(q)\, e_\lambda^\dagger(q'); \qquad \mathcal{P}^-(q,\,q') = \sum_{\lambda<0} e_\lambda(q)\, e_\lambda^\dagger(q'). \tag{9}$$

Let $\mathcal{I}$ be the unity operator on the function space spanned by e_λ. Then, obviously,

$$\mathcal{P}^+ + \mathcal{P}^- = \mathcal{I}. \tag{10}$$

If we join two-dimensional projectors $\mathcal{P}^+$ and $\mathcal{P}^-$ into 4×4 matrix $\mathcal{P}$ the spectral boundary condition for 4-spinor ψ will look as follows:

$$\mathcal{P}\,\psi|_{\partial B_4} = \left(\begin{array}{cc} \mathcal{P}^+ & 0 \\ 0 & \mathcal{P}^- \end{array}\right)\left(\begin{array}{c} u \\ v \end{array}\right)\Bigg|_{\partial B_4} = 0. \tag{11}$$

The projector $\mathcal{P}$ commutes with matrix γ^5:

$$\left[\mathcal{P},\,\gamma^5\right] = 0. \tag{12}$$

Therefore boundary condition (11) by construction respects chiral invariance.

1.2.2. The physics Now we shall prove that the spectral boundary conditions are acceptable and explain their physical meaning. Namely, we shall show that SBC provide Hermicity of the Dirac operator and conservation of fermions in the bag. After that we will explain the origin of requirements (8).

First let us prove that Dirac operator is Hermitian. As usually, we integrate by parts the expression

$$\int_{B_4} dV\, f^\dagger\,(-i\nabla\!\!\!/ g) = \int_{B_4} dV\,(-i\nabla\!\!\!/ f)^\dagger\, g + \oint_{\partial B_4} dS\, f^\dagger\,(-i\gamma^\xi)g. \tag{13}$$

Now we need to show that if f and g satisfy (8) then the last term vanishes.

Conditions (8) mean that on the boundary 4-spinors f and g may be written as: $f = \left(\begin{array}{c} f^- \\ f^+ \end{array}\right)$ and $g = \left(\begin{array}{c} g^- \\ g^+ \end{array}\right)$, where $f^\pm$ and $g^\pm$ include only components with positive and negative λ respectively, see (7). Rewriting the boundary term in (13) we get

$$\oint_{\partial B_4} dS\, f^\dagger\,(-i\gamma^\xi)g = \oint_{\partial B_4} dS\,\left[(f^-)^\dagger\, g^+ - (f^+)^\dagger\, g^-\right] = 0, \tag{14}$$

due to the orthogonality of eigenfunctions of the boundary operator. Thus the APS boundary conditions indeed ensure the Hermicity of Dirac operator.

In addition, relation (14) guarantees conservation of fermions in the bag. Indeed, for $f = g$ the LHS is nothing but the net fermionic current through the boundary,

$$\oint_{\partial B_4} dS\, j^\xi = -i\oint_{\partial B_4} dS\, f^\dagger\gamma^\xi f = 0. \tag{15}$$

Therefore the number of fermions is conserved and particles in the spectral bag are confined.

In order to understand the physics of SBC let us rewrite the eigenvalue condition (5) near the boundary in terms of components.

$$(\partial_\xi + \lambda)\, g^\lambda_\Lambda(\xi) \quad = \Lambda\, f^\lambda_\Lambda(\xi); \tag{16a}$$
$$-(\partial_\xi - \lambda)\, f^\lambda_\Lambda(\xi) = \Lambda\, g^\lambda_\Lambda(\xi). \tag{16b}$$

Depending on the sign of λ these relations reduce on the boundary either to

$$\left.\frac{\partial_\xi g^\lambda_\Lambda}{g^\lambda_\Lambda}\right|_{\xi=0} = -\lambda < 0, \qquad f^\lambda_\Lambda(0) = 0 \quad \text{at} \quad \lambda > 0; \tag{17a}$$

or to

$$\left.\frac{\partial_\xi f^\lambda_\Lambda}{f^\lambda_\Lambda}\right|_{\xi=0} = \quad \lambda < 0, \qquad g^\lambda_\Lambda(0) = 0 \quad \text{at} \quad \lambda < 0. \tag{17b}$$

Thus both components either vanish on the boundary or have a negative logarithmic derivative along the normal.

This requirement has a simple physical interpretation. Suppose that out of the bag the metric and the gauge field remain the same as on the boundary. Then we can continue the functions f and g outside the bag to $\xi = \infty$. Some of the functions will be zero, $f^+ = g^- = 0$ at $\xi > 0$, and the rest will be falling square integrable exponents f^λ, $g^\lambda \propto \exp -|\lambda|\xi$ similar to wave functions of particles locked in a potential well. The only difference is that now the depth of the well depends on λ and is adjusted for each mode specially. We may conclude that the spectral boundary conditions claim that wave functions in the bag must have square integrable continuation to infinity.

2. The SBC for physical bags

2.1. The truncated SBC

Now let us turn to fermions confined in a 3-dimensional spatial bag B_3 that evolves in Euclidean time and sweeps the infinite space-time cylinder $B_3 \otimes R$. We will call the first three coordinates "space" and the fourth one "time". The boundary operator consists of spatial and temporal parts:

$$-\, i\hat{\nabla}_{\partial B_3 \otimes R} = -i\hat{\nabla}_{\partial B_3} - i\sigma^z \partial_4. \tag{18}$$

We will call the spatial part $-i\hat{\nabla}_{\partial B_3}$ the **truncated boundary operator**. Let its eigenfunctions be $e^\pm_\lambda$:

$$-\, i\hat{\nabla}_{\partial B_3}\, e^\pm_\lambda(q) = \pm\lambda\, e^\pm_\lambda(q), \qquad \lambda > 0. \tag{19}$$

Wave functions on the space-time boundary $\partial B_3 \otimes R$ can be expanded in $e^\pm_\lambda$ and longitudinal (temporal) plane waves:

$$u_\Lambda = \sum_{\lambda>0} \int \frac{dk}{2\pi}\, e^{ikt} \left[f^{+\lambda,\,k}_\Lambda\, e^+_\lambda + f^{-\lambda,\,k}_\Lambda\, e^-_\lambda \right]; \tag{20a}$$

$$v_\Lambda = \sum_{\lambda>0} \int \frac{dk}{2\pi}\, e^{ikt} \left[g^{+\lambda,\,k}_\Lambda\, e^+_\lambda + g^{-\lambda,\,k}_\Lambda\, e^-_\lambda \right]. \tag{20b}$$

The truncated operator $-i\hat{\nabla}_{\partial B_3}$ anticommutes with σ^z. Therefore σ^z changes the sign of e-eigenvalues. A possible choice of eigenvectors is (see [7, 8] for the sphere)

$$e^{\pm}_{\lambda} = \pm i\sigma^z \, e^{\mp}_{\lambda}. \tag{21}$$

Thus the last term in (18) mixes positive and negative spatial harmonics.

In classical approach this would mean that SBC should be written in terms of k-dependent eigenfunctions of the full boundary operator (18) which look rather complicated. Moreover, extending boundary conditions onto the entire interval $-\infty < t < \infty$ makes them "future-sensitive" and violates causality. Therefore we propose to consider the k-independent **truncated APS constraints**:

$$f^{+\lambda,\,k}_{\Lambda}\Big|_{\partial B_3} = 0; \tag{22a}$$

$$g^{-\lambda,\,k}_{\Lambda}\Big|_{\partial B_3} = 0. \tag{22b}$$

These conditions do not depend on time and allow Hamiltonian treatment of the system. Besides, they may be applied both in Euclidean and Minkowski spaces. Now let us show that they are acceptable.

2.2. Consistency

We are going to prove that the truncated form of SBC fulfills the necessary conditions. Namely, they are chirally invariant, the Dirac operator is Hermitian, the fermion number is conserved and, after all, wave functions may be continued out of the bag to spatial infinity.

The proof of the first three points literally follows the 4-dimensional case. Everything that concerns formulae (9–15) remains true for truncated ($_T$) 3-dimensional SBC (22). One may define on ∂B_3 projectors,

$$\mathcal{P}^{\pm}_{T}(q,\,q') = \sum_{\lambda>0} e^{\pm}_{\lambda}(q)\,\left[e^{\pm}_{\lambda}(q')\right]^{\dagger}. \tag{23}$$

Then the truncated boundary conditions may be written in the manifestly γ^5-invariant form,

$$\mathcal{P}_T\,\psi|_{\partial B_3} = \begin{pmatrix} \mathcal{P}^{+}_{T} & 0 \\ 0 & \mathcal{P}^{-}_{T} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}\Bigg|_{\partial B_3} = 0. \tag{24}$$

Hermicity of Dirac operator and conservation of fermions are proven in the same way as before, see (13–15), so we skip the formulae.

The last point is more delicate. We already mentioned that the σ^z-piece in (18) mixes positive and negative harmonics. Therefore they must be analysed together and instead of two eigenvalue equations (16) we get four (ξ is the spatial normal to the boundary):

$$(\partial_\xi + \lambda)\, g^{+\lambda,\,k}_{\Lambda} = \Lambda\, f^{+\lambda,\,k}_{\Lambda} + ik\, g^{-\lambda,\,k}_{\Lambda}; \tag{25a}$$

$$-(\partial_\xi - \lambda)\, f^{+\lambda,\,k}_{\Lambda} = \Lambda\, g^{+\lambda,\,k}_{\Lambda} + ik\, f^{-\lambda,\,k}_{\Lambda}\ : \tag{25b}$$

$$(\partial_\xi - \lambda)\, g_\Lambda^{-\lambda,\,k} \quad = \Lambda\, f_\Lambda^{-\lambda,\,k} - ik\, g_\Lambda^{+\lambda,\,k}; \tag{25c}$$

$$-(\partial_\xi + \lambda)\, f_\Lambda^{-\lambda,\,k} = \Lambda\, g_\Lambda^{-\lambda,\,k} - ik\, f_\Lambda^{+\lambda,\,k}. \tag{25d}$$

The new feature with respect to (16) are ik-terms that appear due to the mixing. However one may notice that the terms in the RHS of (25) come in pairs f^+, g^- and f^-, g^+. Therefore according to conditions (22) the RHS of equations (25a, 25d) still vanish on the boundary. Thus the behaviour of g^+ and f^- on the boundary is governed by the homogeneous equations and

$$\left.\frac{\partial_\xi f_\Lambda^{-\lambda,\,k}}{f_\Lambda^{-\lambda,\,k}}\right|_{\xi=0} = \left.\frac{\partial_\xi g_\Lambda^{+\lambda,\,k}}{g_\Lambda^{+\lambda,\,k}}\right|_{\xi=0} = -\lambda < 0. \tag{26}$$

Hence despite the presence of extra pieces the nonvanishing components g^+ and f^- have negative logarithmic derivatives. This means that solutions of eigenvalue equations may be continued from the world cylinder swept by evolving bag to spatial infinity in an integrable way. Thus the last of the requirements is fulfilled. This completes the proof of acceptability of the truncated SBC.

Conclusion

The truncated version of APS boundary conditions offers a number of possibilities. It allows to formulate a chirally invariant bag model and to address chiral properties of fermionic field in the closed volume. The constraints do not depend on time so one may write down the Hamiltonian and study the energy (and mass) spectrum of the system. Another advantage is that the modified SBC may be used both in Euclidean and Minkowski spaces.

A new feature that SBC may bring to bag physics is their nonlocality. The usually employed local boundary conditions, see [1, 2, 3], correspond to the thin wall approximation. The spectral conditions refer to the boundary as a whole. Therefore, in a sense, hadrons are also treated as a whole which complies with modern concepts. It would be interesting to investigate hadronic spectra in chiral invariant bags and find out if the model is realistic and what it is missing.

Another question is more mathematical. Chiral symmetry is specific for fermions in even-dimensional spaces. Hence the spectral boundary conditions were always discussed in even dimensions. The truncated SBC are formulated in the odd-dimensional space that remains after discarding the time. This might have interesting consequences. For example, the boundary of odd-dimensional bag is an even-dimensional manyfold and the truncated boundary operator possesses a sort of internal chirality. It would be interesting to study consequences of this hidden symmetry.

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